

Torsion, an alternative to the cosmological constant?

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Abstract

We confront Einstein-Cartan's theory with the Hubble diagram and obtain a negative answer to the question in the title. Contrary findings in the literature seem to stem from an error in the field equations.

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1 Introduction

A recent fit [1] of Einstein-Cartan's theory [2, 3, 4] to the Hubble diagram of supernovae was rather encouraging in that the spin-density, which is the source of torsion, could – within today's error-bars – replace cold dark matter. The underlying space was flat, homogeneous, isotropic and invariant under inversion. In the present work we extend our fit by admitting parity violation. Our motivation is that weak forces do break parity. Parity odd Einstein-Cartan theory in the context of maximally symmetric cosmology has a long history. Already in 1978, Bloomer [5] analyzed the theory, with the parity odd part only, on the 3-sphere. In 1986, Peter Minkowski [6] reconsidered it on flat \mathbb{R}^3 . He mentions the handedness of spiral galaxies as motivation. More recently, in 2002 Capozziello *et al.* [7] took up the flat, parity odd theory again. They claim that spin-density can replace the cosmological constant.

2 Notations and field equations

We use the conventions of reference [1]. For the reader's convenience we briefly summarize them.

Let x^μ ($\mu = 0, 1, 2, 3$) be a coordinate system on an open subset of \mathbb{R}^4 . We will use both a holonomic frame dx^μ and an orthonormal frame, *un repère mobile* using Cartan's words, $e^a =: e^a_\mu dx^\mu$, $a = 0, 1, 2, 3$. We denote a metric connection with respect to an orthonormal frame by $\omega^a_b =: \omega^a_{bc} e^c$. It is as a 1-form with values in the Lie algebra of the Lorentz group. We follow the traditional convention and denote the same connection with respect to the holonomic frame by a different letter: $\Gamma^\alpha_\beta =: \Gamma^\alpha_{\beta\mu} dx^\mu$, a $gl(4)$ valued 1-form. The link between the components of the connection with respect to the holonomic frame Γ and with respect to the orthonormal frame ω is given by the $GL(4)$ gauge transformation with $e(x) = e^a_\mu(x) \in GL(4)$;

$$\omega = e\Gamma e^{-1} + ede^{-1}. \quad (1)$$

Then (suppressing all wedge symbols) Cartan's two structure equations read:

$$R := d\omega + \frac{1}{2}[\omega, \omega], \quad (2)$$

for the curvature 2-form $R^a_b =: \frac{1}{2}R^a_{bcd}e^ce^d$, and

$$T := De = de + \omega e, \quad (3)$$

for the torsion 2-form, $T^a =: \frac{1}{2}T^a_{bc}e^be^c$. It will be useful to decompose the torsion tensor into its three irreducible parts:

$$T_{abc} = A_{abc} + \eta_{ab}V_c - \eta_{ac}V_b + M_{abc}, \quad (4)$$

with the completely antisymmetric part $A_{abc} := \frac{1}{3}(T_{abc} + T_{cab} + T_{bca})$, the vector part $V_c := \frac{1}{3}T_{abc}\eta^{ab}$, and the mixed part M_{abc} characterized by $M_{abc} = -M_{acb}$, $M_{abc}\eta^{ab} = 0$, and $M_{abc} + M_{cab} + M_{bca} = 0$.

In a Riemann-Cartan space there are two kinds of geodesics: curves that minimize the arc-length (or proper time) with respect to the metric and curves whose tangent vectors are parallel with respect to the connection. These two geodesics coincide if and only if the torsion only has a completely antisymmetric part, $V = 0$, $M = 0$.

With these notations, the Einstein-Hilbert action reads

$$\begin{aligned} S_{\text{EH}}[e, \omega] &= \frac{-1}{32\pi G} \int (R^{ab} + \frac{1}{6}\Lambda e^a e^b) e^c e^d \epsilon_{abcd} \\ &= \frac{-1}{16\pi G} \int (R^{ab}{}_{ab} + 2\Lambda) dV, \end{aligned} \quad (5)$$

where $\epsilon_{0123} = 1$. The energy-momentum current is the vector-valued 3-form τ_a obtained by varying the orthonormal frame in the matter Lagrangian:

$$\mathcal{L}_M[e + f, \omega] - \mathcal{L}_M[e, \omega] =: -f^a \tau_a + O(f^2). \quad (6)$$

The energy-momentum tensor τ_{ab} is defined by $*\tau_a =: \tau_{ab} e^b$.

Likewise, the spin current is the Lorentz-valued 3-form S_{ab} obtained by varying the connection in the matter Lagrangian:

$$\mathcal{L}_M[e, \omega + \chi] - \mathcal{L}_M[e, \omega] =: -\frac{1}{2}\chi^{ab} S_{ab} + O(\chi^2). \quad (7)$$

The spin tensor S_{abc} is defined by $*S_{ab} =: S_{abc} e^c$, where $*$ is the Hodge star of the metric. Einstein's equations are obtained by varying the total action with respect to the orthonormal frame:

$$(R^{ab} + \frac{1}{3}\Lambda e^a e^b) e^d \epsilon_{abcd} = -16\pi G \tau_c \quad \text{or equivalently} \quad G_{ab} - \Lambda \eta_{ab} = 8\pi G \tau_{ab}, \quad (8)$$

with the Einstein tensor $G_{ab} := R^c{}_{acb} - \frac{1}{2}R^{cd}{}_{cd} \eta_{ab}$.

Likewise Cartan's equations are derived by varying the total action with respect to the connection:

$$T^c e^d \epsilon_{abcd} = -8\pi G S_{ab}, \quad (9)$$

or equivalently:

$$A_{cab} + 3V_a \eta_{bc} - 3V_b \eta_{ac} + M_{cab} = -8\pi G S_{abc}. \quad (10)$$

3 Homogeneous and isotropic spaces

The invariance of the metric tensor $g_{\mu\nu}(x) = e^a{}_\mu(x) e^b{}_\nu(x) \eta_{ab}$ under an infinitesimal diffeomorphism ξ is expressed by the Killing equation:

$$\xi^\alpha \frac{\partial}{\partial x^\alpha} g_{\mu\nu} + \frac{\partial \xi^{\bar{\mu}}}{\partial x^\mu} g_{\bar{\mu}\nu} + \frac{\partial \xi^{\bar{\nu}}}{\partial x^\nu} g_{\mu\bar{\nu}} = 0. \quad (11)$$

Likewise the vector field ξ preserves the connection if

$$\xi^\alpha \frac{\partial}{\partial x^\alpha} \Gamma^\lambda{}_{\mu\nu} - \frac{\partial \xi^\lambda}{\partial x^\lambda} \Gamma^{\bar{\lambda}}{}_{\mu\nu} + \frac{\partial \xi^{\bar{\mu}}}{\partial x^\mu} \Gamma^\lambda{}_{\bar{\mu}\nu} + \frac{\partial \xi^{\bar{\nu}}}{\partial x^\nu} \Gamma^\lambda{}_{\mu\bar{\nu}} + \frac{\partial^2 \xi^\lambda}{\partial x^\mu \partial x^\nu} = 0. \quad (12)$$

The most general Riemann-Cartan space invariant under $SO(3) \ltimes \mathbb{R}^3$ has the Robertson-Walker metric: $e^0 = dt$, $e^1 = a dx$, $e^2 = a dy$, $e^3 = a dz$, with the scale factor $a(t)$, a positive function of cosmic time t . The non-vanishing components ω^a_{bc} of the most general $SO(3) \ltimes \mathbb{R}^3$ invariant connection are:

$$\omega^0_{ij} = \omega^i_{0j} = \frac{b}{a} \delta_{ij}, \quad \omega^i_{jk} = \frac{f}{a} \epsilon_{ijk}, \quad (13)$$

with two additional functions $b(t)$ and $f(t)$. The first is parity even like the scale factor, the second is parity odd.

The Riemann tensor has the following non-vanishing components:

$$R^0_{i0k} = R^i_{00k} = \frac{b'}{a} \delta_{ik}, \quad R^0_{ijk} = -2 \frac{bf}{a^2} \epsilon_{ijk}, \quad (14)$$

$$R^i_{j0k} = \frac{f'}{a} \epsilon_{ijk}, \quad R^i_{jkl} = \frac{b^2 - f^2}{a^2} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}). \quad (15)$$

The Einstein tensor has:

$$G_{00} = 3 \frac{b^2 - f^2}{a^2}, \quad G_{ij} = - \left(2 \frac{b'}{a} + \frac{b^2 - f^2}{a^2} \right) \delta_{ij}. \quad (16)$$

The torsion tensor has:

$$T^i_{0j} = \frac{a' - b}{a} \delta_{ij}, \quad T_{ijk} = 2 \frac{f}{a} \epsilon_{ijk}. \quad (17)$$

The antisymmetric part has only space components, $A_{ijk} = 2f/a \epsilon_{ijk}$, the vector part has only a time component, $V_0 = (b - a')/a$, and the mixed part vanishes, $M = 0$. This result agrees with the curvature and torsion found in references [8, 9] for spacetimes with maximally symmetric 3-spaces.

4 Equations of state and Friedmann equations

The most general $SO(3) \ltimes \mathbb{R}^3$ -invariant energy-momentum tensor contains two function of time, the energy density $\rho(t)$ and the pressure $p(t)$ and one usually assumes an equation of state $p(t) =: w \rho(t)$.

Likewise the most general $SO(3) \ltimes \mathbb{R}^3$ -invariant spin density has two functions of time $s(t)$ and $\tilde{s}(t)$ in the two irreducible components: $S_{0jk} =: -s(t) \delta_{jk}$ and $S_{ijk} =: -\tilde{s}(t) \epsilon_{ijk}$ and we assume two equations of state:

$$s(t) =: w_s \rho(t), \quad \tilde{s}(t) =: w_{\tilde{s}} \rho(t). \quad (18)$$

Then the generalised Friedmann equations, i.e. the tt and the xx components of Einstein's

equations, and Cartan's equations read:

$$3 \frac{b^2 - f^2}{a^2} = \Lambda + 8\pi G\rho, \quad (19)$$

$$2 \frac{b'}{a} + \frac{b^2 - f^2}{a^2} = \Lambda - 8\pi Gp, \quad (20)$$

$$3 \frac{a' - b}{a} = 8\pi Gw_s\rho \quad (21)$$

$$2 \frac{f}{a} = 8\pi Gw_{\bar{s}}\rho. \quad (22)$$

These equations agree with results in references [8, 9] and in reference [5] except for a missing 1/3 in front of the last term on the right-hand side of equation (23) there. Note that this factor re-appears correctly in the subsequent equation (24).

However we disagree with a result in reference [6] stating that the field equations imply that the function $f(t)$ (in our notations) must be constant. This result is reproduced in reference [7], probably because of a missing factor 3 in its equation (20) (in the arXiv version) and it is only with this factor missing that the presumably constant f can be interpreted as a cosmological constant.

Putting the pressure to zero, $p = 0$, we have four equations for four unknown functions: a , b , f and ρ . Equations (19) and (22) are algebraic, the other two equations, (20) and (21), are first order differential equations for a and b . We use the algebraic ones to eliminate ρ and f . Then we have a unique solution with two initial conditions $a(0) = a_0$ and $b(0) = b_0$. We therefore have five parameters, a_0 , b_0 , Λ , w_s and $w_{\bar{s}}$. (We assume Newton's constant known.) These five parameters then fix $\rho(0)$ by use of equation (19):

$$1 = \Omega_{m0} + \Omega_{\Lambda0} + 2\Omega_{s0} - \Omega_{s0}^2 + \frac{9}{4}\Omega_{\bar{s}0}^2, \quad (23)$$

with familiar dimensionless quantities:

$$\Omega_m := \frac{8\pi G\rho}{3H^2}, \quad \Omega_{\Lambda} := \frac{\Lambda}{3H^2}, \quad \Omega_s := w_s H \frac{8\pi G\rho}{3H^2}, \quad \Omega_{\bar{s}} := w_{\bar{s}} H \frac{8\pi G\rho}{3H^2}. \quad (24)$$

In particular, we see that the scale factor today a_0 has dropped out. This is well-known for cosmology with vanishing spatial curvature and remains true in presence of non-vanishing torsion. Note also that the sign of f does not matter because only its square appears in the two Einstein equations. Therefore we may assume the state parameter $w_{\bar{s}}$ to be non-negative.

5 Hubble diagram

To compute the Hubble diagram, we must solve the geodesic equations for co-moving galaxies and for photons [10]. For both, torsion decouples and they reduce to the geodesic equations with the Christoffel connection of the metric. Consequently the redshift is still

given by $z = a_0/a(t) - 1$ and the apparent luminosity ℓ is still related to the absolute luminosity of the standard candle L by

$$\ell(t) = \frac{L}{4\pi a_0^2 x(t)^2} \frac{a(t)^2}{a_0^2}. \quad (25)$$

We have put the earth at the origin of the Cartesian coordinates and the supernova on the x -axis:

$$x(t) := \int_t^{t_0} \frac{d\tilde{t}}{a(\tilde{t})}. \quad (26)$$

Note that the Einstein equations in presence of half-integer spin do feel torsion. However the link between the Hubble constant H_0 and $d(z^2\ell)/dz(0)$ is purely kinematical and therefore does not depend on torsion. This fact will be crucial to identify consistently the initial conditions of Friedmann's equations.

6 Data analysis

The data analysis used in this paper has been fully described in [1]. Only a brief reminder is given here. The type 1a supernovae Hubble diagram is constructed using the Union 2 sample [11] with 557 supernovae and the full systematic error matrix. The magnitude of supernovae is written as $M(z) = m_s + 2.5 \log \ell(z)$ where m_s is a normalization parameter fitted to the data and $\ell(z)$ the apparent luminosity (25).

The apparent luminosity is computed using the generalized Friedmann equations (19), (20), (21) and (22). These equations are solved numerically by using the Runge-Kutta algorithm [12].

The MINUIT [14] package is used to fit the best cosmology by minimizing the χ^2 defined as:

$$\chi^2 = \Delta M^T V^{-1} \Delta M, \quad (27)$$

where ΔM is the vector of differences between measured and expected magnitudes and V the full covariance matrix including systematic errors.

Marginalization over unwanted parameters as m_s and error estimates or contour constructions are obtained using the frequentist prescription [15]. The Einstein-Cartan cosmology fit is performed with 3 or 4 free parameters ($m_s, \Omega_m, \Omega_s, \Omega_{\bar{s}}$) while Ω_Λ is derived from the Friedmann-like equation (23).

Table 1 presents the results of the fit of Einstein-Cartan's theory (parity even and/or odd) and, for comparison, the results of the fit of the pure Einstein theory in the first line. Because of very high non-Gaussianity, errors are given at 1 and 2 sigma level. Minimum χ^2 for all theories are statistically equivalent. If in the parity even case the preferred value for Ω_m is compatible at a level of one sigma with baryon matter density, in the odd-parity case, the preferred value of Ω_m is in agreement with the total matter density of 0.27 published by the WMAP collaboration [13]. This is not surprising since the preferred

| | Ω_m | Ω_Λ | Ω_s | $\Omega_{\tilde{s}}$ | χ^2_{min} |
|-------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------|
| Einstein | $0.35^{+0.10+0.15}_{-0.11-0.17}$ | $0.88^{+0.19+0.28}_{-0.11-0.32}$ | 0. | 0. | 530.0 |
| even-parity torsion | $0.09^{+0.30+0.47}_{-0.07-0.08}$ | $0.83^{+0.10+0.12}_{-0.16-0.23}$ | $0.04^{+0.01+0.02}_{-0.07-0.12}$ | 0. | 530.4 |
| odd-parity torsion | $0.27^{+0.03+0.06}_{-0.02-0.27}$ | $0.73^{+0.04+0.06}_{-0.11-0.32}$ | 0. | $0.^{+0.22+0.55}_{-0.22-0.55}$ | 530.4 |
| odd-even parity | $0.08^{+0.27+0.9}_{-0.07-0.08}$ | $0.85^{+0.10+0.15}_{-0.15-0.25}$ | $0.04^{+0.02+0.06}_{-0.06-0.34}$ | $0.^{+0.01+0.6}_{-0.01-0.6}$ | 530.0 |
| odd parity no Λ | $0.01^{+0.02+0.03}_{-0.02-0.03}$ | 0. | 0. | $0.66^{+0.01+0.02}_{-0.01-0.02}$ | 560.7 |

Table 1: Fit results (1 and 2σ errors) for Einstein and Einstein-Cartan theories with even and odd parity. No flatness constraint is imposed in the pure Einstein's theory.

value of $\Omega_{\tilde{s}}$ is exactly zero implying a flat Universe in the pure Einstein theory. In all cases, the cosmological constant energy density is only slightly changed.

In figure 1(a) the result of the Hubble diagram fit with odd Einstein-Cartan theory is shown (upper curve) with data points and error bars. As in the case of parity even Einstein-Cartan theory, the agreement between fitted curve and data points is excellent. The lower curve shows the fit resulting from putting the cosmological constant to zero for parity odd torsion. The agreement with data points seems good and suggests that the cosmological constant can be replaced by parity odd torsion.

To test this hypothesis quantitatively, we use the log likelihood ratio technique. The log likelihood ratio is defined as:

$$R = -2Ln \frac{Sup(\mathcal{L}(m_s, \Omega_{\tilde{s}}, \Omega_\Lambda = 0))}{Sup(\mathcal{L}(m_s, \Omega_{\tilde{s}}, \Omega_\Lambda))}. \quad (28)$$

Here Sup denotes the supremum of the likelihood function defined in term of the χ^2 :

$$\mathcal{L} = \frac{1}{(2\pi)^{(n/2)}|V|^{1/2}} e^{(-\frac{\chi^2}{2})}, \quad (29)$$

n is the number of data points and V the full error matrix. Thus the log likelihood ratio reads simply:

$$R = \chi^2_{min,1} - \chi^2_{min,2}. \quad (30)$$

The probability distribution of this test variable is approximately a χ^2 distribution with degree of freedom equal to the difference between the degrees of freedom of both models, one in this case.

The minimum χ^2 for the null cosmological constant hypothesis is equal to 560.7 while the minimum χ^2 for odd parity torsion with cosmological constant is 530.4 (Table 1). The p -value is found to be equal to $6 \cdot 10^{-8}$ corresponding to 5.4σ significance.

Because systematic errors of supernovae intrinsic magnitude variations at high redshift (above 1) can be important, we check the null cosmological constant hypothesis using only supernovae at redshift below 1. The log likelihood ratio is found to be 25 leading to a p -value of $5.3 \cdot 10^{-7}$ or a significance of 5.01 sigma. Thus, the null cosmological constant hypothesis within the parity odd Einstein-Cartan theory is ruled out at more than 5σ .

For completeness, we perform the same analysis using simultaneously even and odd parity torsion. The resulting fit is slightly better because of one more fitted parameter. The χ^2 of the fit is equal to 560.1 leading to a p -value of 5.9610^{-7} corresponding to a 5.4σ significance.

7 Conclusion

We find that a fit of Einstein-Cartan's theory to the Hubble diagram is incompatible at 5σ level with the replacement of the cosmological constant by torsion, parity preserving or not. We think that the contrary claim in reference [7] relies on a wrong coefficient in the field equations [6, 7].

References

- [1] A. Tilquin and T. Schücker, *Torsion, an alternative to dark matter?*, arXiv:1104.0160 [astro-ph.CO], Gen. Rel. Grav. in press.
- [2] É. Cartan, *Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie)*, Ann. Éc. Norm. Sup. **40** (1923) 325.
(première partie, suite), Ann. Éc. Norm. Sup. **41** (1924) 1.
(deuxième partie), Ann. Éc. Norm. Sup. **42** (1925) 17.
- [3] A historic review is:
F. W. Hehl, P. von der Heyde, G. D. Kerlick and J. M. Nester, *General relativity with spin and torsion: Foundations and prospects*, Rev. Mod. Phys. **48** (1978) 393.
- [4] Two recent reviews are:
S. Capozziello, G. Lambiase, C. Stornaiolo, *Geometric classification of the torsion tensor in space-time*, Annalen Phys. **10** (2001) 713. [gr-qc/0101038].
I. L. Shapiro, *Physical aspects of the space-time torsion*, Phys. Rept. **357** (2002) 113. [hep-th/0103093].
- [5] I. Bloomer, *A maximally symmetric space with torsion*, Gen. Rel. Grav. **9** (1978) 763.
- [6] P. Minkowski, *On the cosmological equations in the presence of a spatially homogeneous torsion field*, Phys. Lett. B **173** (1986) 247.
- [7] S. Capozziello, V. F. Cardone, E. Piedipalumbo, M. Sereno, A. Troisi, *Matching torsion Lambda - term with observations*, Int. J. Mod. Phys. **D12** (2003) 381. [astro-ph/0209610].
- [8] M. Tsamparlis, *Cosmological principle and torsion*, Phys. Lett. A **75** (1979) 27.
M. Tsamparlis, *Methods For Deriving Solutions In Generalized Theories Of Gravitation: The Einstein-Cartan Theory*, Phys. Rev. **D24** (1981) 1451.

- [9] H. Goenner, F. Müller-Hoissen, *Spatially homogeneous and isotropic space in theories of gravitation with torsion*, Class. Quant. Grav. **1** (1984) 651.
- [10] see for example M. Berry, **Principles of Cosmology and Gravitation** (1976) Cambridge University Press.
- [11] R. Amanullah, et al., *Spectra and HST Light Curves of Six Type IA Supernovae at $0.511 < z < 1.12$ and the Union2 Compilation*; ApJ: April 9, 2010
- [12] L.F. Shampine and H.A. Watts, *The Art of Writing a Runge-Kutta Code, Part I*, **Mathematical Software III** (1979), JR Rice, ed. New York: Academic Press, p. 257
The Art of Writing a Runge-Kutta Code, Part II, Applied Mathematics and Computation **5** p.93.
- [13] D. Larson et al., *Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Power Spectra and WMAP Derived Parameters*; ApJ. Suppl. **192** (2011) 16.
- [14] *The ROOT analysis package*, <http://root.cern.ch/drupal/>
- [15] C. Amsler et al. *Review of Particle Physics*. Phys. Lett. **B** (2008) 667.

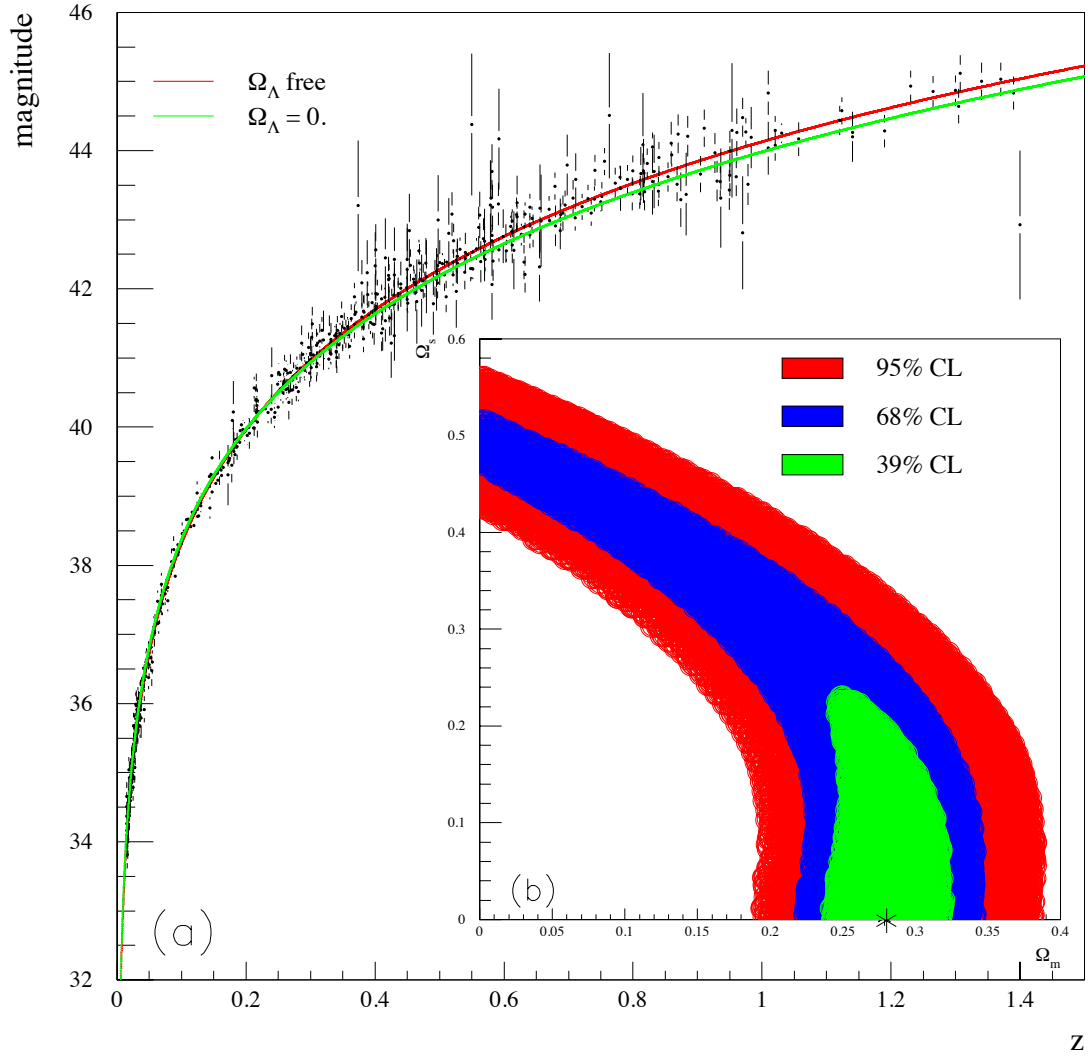


Figure 1: (a) Fit results using the Union 2 Supernovae sample. The red (upper) curve corresponds to the Einstein-Cartan 3-fit $(m_s, \Omega_m, \Omega_{\tilde{s}})$ while the green (lower) curve represents the 2-fit assuming a vanishing cosmological constant. (b) 39%, 68% and 95% confidence level contour in the $(\Omega_m, \Omega_{\tilde{s}})$ plane for the Einstein-Cartan 3-fit.